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A resource theory of entanglement with a unique multipartite maximally entangled state

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Entanglement theory is formulated as a quantum resource theory in which the free operations are local operations and classical communication (LOCC). This defines a partial order among bipartite pure states that makes it possible to identify a maximally entangled state, which turns out to be the most relevant state in applications. However, the situation changes drastically in the multipartite regime. Not only do there exist inequivalent forms of entanglement forbidding the existence of a unique maximally entangled state, but recent results have shown that LOCC induces a trivial ordering: almost all pure entangled multipartite states are incomparable (i.e. LOCC transformations among them are almost never possible). In order to cope with this problem we consider alternative resource theories in which we relax the class of LOCC to operations that do not create entanglement. We consider two possible theories depending on whether resources correspond to multipartite entangled or genuinely multipartite entangled (GME) states and we show that they are both non-trivial: no inequivalent forms of entanglement exist in them and they induce a meaningful partial order (i.e. every pure state is transformable to more weakly entangled pure states). Moreover, we prove that the resource theory of GME that we formulate here has a unique maximally entangled state, the generalized GHZ state, which can be transformed to any other state by the allowed free operations.

Introduction. Entanglement is a striking feature of quantum theory with no classical analogue. Although initially studied to address foundational issues [1], the development of quantum information theory [2] in the last few decades has elevated it to a resource that allows tasks to be implemented which are impossible in classical systems. The resource theory of entanglement [3] aims at providing a rigorous framework to qualify and quantify entanglement and, ultimately, to understand fully its capabilities and limitations within the realm of quantum technologies. However, this theory is much more firmly developed for bipartite than multipartite systems. In fact, although a few applications have been proposed within the latter setting such as secret sharing [4], the one-way quantum computer [5] and metrology [6], a deeper understanding of the complex structure of multipartite entangled states might inspire further protocols in quantum information science and better tools for the study of condensed-matter systems.

The wide applicability of the formulation of entanglement theory as a resource theory has motivated an active line of work [7] that studies different quantum effects from this point of view such as coherence [8], reference frame alignment [9], thermodynamics [10], non-locality [11] or steering [12]. The main question a resource theory addresses is to order the set of states and provide means to quantify their nature as a resource. The so-called free operations are crucial to this task. This is a subset of transformations, which the given scenario dictates can be implemented at no cost. Thus, all states that can be prepared with these operations are free states. Conversely, non-free states acquire the status of a resource: granted such states, the limitations of the corresponding scenario might be overcome. Moreover, the concept of free operations allows an order relation to be defined. If a state ρ can be transformed into σ by some free operation, then ρ cannot be less resourceful than σ since any task achievable by σ is also achievable by ρ as the corresponding transformation can be freely implemented. However, the converse is not necessarily true. Furthermore, one can introduce resource quantifiers as functionals that preserve this order.

Since entanglement is a property of systems with many constituents which may be far away, the natural choice for free operations in this resource theory is local operations and classical communication (LOCC). Indeed, parties bound to LOCC can only prepare separable states, and entangled states become a resource to overcome the constraints imposed by LOCC manipulation. Nielsen characterized in [13] the possible LOCC conversions among pure bipartite states, which revealed that the LOCC ordering reduces to majorization [14] and, remarkably, that there is a unique maximally entangled state for fixed local dimension. This is because this state can be transformed by LOCC into any other state of that dimension but no other state of that dimension can be transformed into it. This state is then regarded as a gold standard to measure entanglement and, unsurprisingly, it turns out to be the most useful state for bipartite entanglement applications such as teleportation. Importantly, the situation changes drastically in the multipartite case. Here, reference [15] and subsequent work [16] have shown that there exist inequivalent forms of entanglement: the state space is divided into classes, the so-called stochastic

LOCC (SLOCC) classes, of states which can be interconverted with non-zero probability by LOCC but cannot be transformed outside the class by LOCC, even probabilistically. This in particular shows that no maximally entangled state can exist for multipartite states. Still, one could in principle study the ordering induced by LOCC within each SLOCC class. Recent work [17] in this direction has revealed, however, an extreme feature that culminates with the result of Ref. [18]: almost all pure states of more than three parties are *isolated*, i.e. they cannot be obtained from nor transformed to another inequivalent pure state of the same local dimensions by LOCC. This means that almost all pure states are incomparable by LOCC, inducing a trivial ordering and a meaningless arbitrariness in the construction of entanglement measures. In this sense, one may say that the resource theory of multipartite entanglement with LOCC is generically trivial.

We believe this calls for a critical reexamination of the resource theory of entanglement and, in particular, for the notion of LOCC as the ordering-defining relation. Indeed, although LOCC transformations have a clear operational interpretation, this is not, in fact, the most general class of transformations that maps the set of separable states into itself. In other words, LOCC is strictly included in the class of non-entangling operations. Thus, from the abstract point of view of resource theories other consistent theories of entanglement (i.e. with separable states being the free states) are possible where the set of free operations is larger than LOCC. Hence, in principle, these could give a more meaningful ordering and revealing structure in the set of multipartite entangled states. To study such possibility is precisely the goal of this Letter. A similar approach has been taken to address other unsatisfying features of the resource theory of entanglement under LOCC such as irreversibility of state transformations for an arbitrarily large number of copies [19]. Remarkably, reference [20] has shown that shifting the paradigm from LOCC to asymptotic non-entangling operations provides a reversible theory of asymptotic entanglement interconversion with a unique entanglement measure and this result has been extended in [21] to arbitrary resource theories under asymptotic resource-non-generating operations [7]. Also, in the absence of a clear set of physical constraints determining the free operations, certain quantum resource theories have been constructed by first defining the set of free states and then considering classes of operations that preserve this set. This is the case of the resource theory of coherence [22], which has been found useful in e.g. metrology applications [23] and quantum channel discrimination [24] and which has subsequently given rise to a fruitful research line considering an operational interpretation for the set of free operations (see [8, 25] and references therein).

Since we seek whether a non-trivial theory is at all possible for single-copy manipulations, we consider here the resource theory of entanglement under the largest possible class of free operations in this regime: strictly non-entangling operations. However, multipartite entanglement comes in two different forms. We will call entangled those states that are not fully separable (FS), while we will call genuinely multipartite entangled (GME) those states which are not biseparable (BS). Thus, one can formulate two theories: one in which entangled states are considered a resource and where the free operations are full separability-preserving (FSP) and the analogous with GME states and biseparability-preserving (BSP) operations. Interestingly, our first result is that both formalisms lead to non-trivial theories: no resource state is isolated in any of these scenarios. Moreover, we show that there are no inequivalent forms of entanglement. Then, we consider whether there exists a unique multipartite maximally entangled state in these theories like in the bipartite case. While we find a negative answer (at least in the simplest non-trivial case of 3-qubit states) for FSP operations, our main result is that the question is answered affirmatively in the resource theory of GME under BSP operations. The maximally GME state turns out to be the generalized Greenberger-Horne-Zeilinger (GHZ) state.

Definitions and preliminaries. We will consider n-partite systems with local dimension d, i.e. states in the Hilbert space $H = H_1 \otimes \cdots \otimes H_n = (\mathbb{C}^d)^{\otimes n}$. Given a subset M of $[n] = \{1, \ldots, n\}$ and its complement $\bar{M}$, we denote by H_M the tensor product of the Hilbert spaces corresponding to the parties in M and analogously with $H_{\bar{M}}$. A pure state $|\psi\rangle \in H$ is FS (otherwise entangled) if $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$ for some states $|\psi_i\rangle \in H_i \ \forall i$, while it is BS (otherwise GME) if $|\psi\rangle = |\psi_M\rangle \otimes |\psi_{\bar{M}}\rangle$ for some states $|\psi_M\rangle \in H_M$ and $|\psi_{\bar{M}}\rangle \in H_{\bar{M}}$ and $M \subsetneq [n]$. These notions are extended to mixed states by the convex hull and we define the sets of FS and BS states by

$$\mathcal{FS} = \mathrm{conv}\{\psi : |\psi\rangle \text{ is FS}\}, \ \mathcal{BS} = \mathrm{conv}\{\psi : |\psi\rangle \text{ is BS}\}, \tag{1}$$

where here and throughout the paper we use the notation $\psi = |\psi\rangle\langle\psi|$ whenever a state is specified as pure. Transformations in quantum theory are given by completely positive and trace preserving (CPTP) maps and we say that such a map Λ (from and to operators on H) is FSP (BSP) if $\Lambda(\rho) \in \mathcal{FS} \ \forall \rho \in \mathcal{FS}$ ($\Lambda(\rho) \in \mathcal{BS} \ \forall \rho \in \mathcal{BS}$). We will say that a functional E taking operators on H to non-negative real numbers is an FSP-measure (BSP-measure) if $E(\rho) \geq E(\Lambda(\rho))$ for every state ρ and FSP (BSP) map Λ. This is completely analogous to entanglement measures, which are required to be non-increasing under LOCC maps. Although LOCC is a strict subset of the FSP and BSP maps, some well-known entanglement measures are still FSP- or BSP-measures and this will play an important role in assessing which transformations are possible within the two formalisms that we consider here. Indeed, measures of the form

$$E_{\mathcal{X}}(\rho)=\inf_{\sigma\in\mathcal{X}}E(\rho||\sigma), \quad (2)$$

where $\mathcal{X}$ stands for either $\mathcal{FS}$ or $\mathcal{BS}$, have the corresponding monotonicity property as long as the distinguishability measure $E(\rho||\sigma)$ is contractive, i.e. $E(\Lambda(\rho)||\Lambda(\sigma))\leq E(\rho||\sigma)$ for every CPTP map Λ. This includes the relative entropy of entanglement [26] for $E(\rho||\sigma)=\mathrm{tr}(\rho\log\rho)-\mathrm{tr}(\rho\log\sigma)$ and the robustness ($R_{\mathcal{X}}$) [27] for

$$E(\rho||\sigma)=R(\rho||\sigma)=\min\{s:(\rho+s\sigma)/(1+s)\in\mathcal{X}\}. \quad (3)$$

If one uses the fidelity $E(\rho||\sigma)=1-F(\rho||\sigma)=1-\mathrm{tr}^2\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$, for pure states Eq. (2) boils down to the geometric measure [28], which we will denote by $G_{\mathcal{X}}$ and which is then seen to be a measure under maps that preserve $\mathcal{X}$. Notice, however, that, as has been recently shown in the bipartite case in [29], not all LOCC-measures remain monotonic under non-entangling maps since the latter formalism allows state conversions that the former does not. In the following, in order to understand the ordering of resources induced by these theories, we study which transformations are possible among pure states under FSP and BSP maps. However, first one should point out that whenever there exist maps Λ and Λ' in the corresponding class of free operations such that $\Lambda(\psi)=\phi$ and $\Lambda'(\phi)=\psi$, then the states ψ and ϕ are equally resourceful and should be regarded as equivalent in the corresponding theory. This is moreover necessary so as to have a well-defined partial order. Hence, although for simplicity we will talk about properties of states, one should have in mind that one is actually speaking about equivalence classes. Specifically, it is known that two pure states are interconvertible by LOCC if and only if they are related by local unitary transformations [30]. Interestingly, we will see that the equivalence classes are wider in the resource theory of GME under BSP. It should be stressed that, to our knowledge, this is the first time that a resource theory of GME is formulated. Notice that the restriction to LOCC can only have FS states as free states. Furthermore, allowing a strict subset of parties to act jointly and classical communication does not fit the bill either as $\mathcal{BS}$ is not closed under these operations.

Non-triviality of the theories. Our first two results are valid in both the FSP and BSP regimes. Thus, following the notation above, the two possible classes of maps will be referred to as $\mathcal{X}$-preserving.

Theorem 1 (collapse of the SLOCC classes). *In a resource theory of entanglement where the free operations are $\mathcal{X}$-preserving maps, all resource states are interconvertible with non-zero probability, i.e. given any pure $\psi_1,\psi_2\notin\mathcal{X}$, there exists a completely positive and trace non-increasing $\mathcal{X}$-preserving map Λ such that $\Lambda(\psi_1)=p\psi_2$ with $p\in(0,1]$.*

Theorem 2 (no isolation). *In a resource theory of entanglement where the free operations are $\mathcal{X}$-preserving maps, no resource state is isolated, i.e. given any pure $\psi_1\notin\mathcal{X}$ on H, there exists an inequivalent pure $\psi_2\notin\mathcal{X}$ on H and a CPTP $\mathcal{X}$-preserving map Λ such that $\Lambda(\psi_1)=\psi_2$.*

The full proof of these two results can be found in [31]. The proof of Theorem 1 is based on explicitly constructing a completely positive and trace non-increasing $\mathcal{X}$-preserving map Λ such that $\Lambda(\psi_1)=p\psi_2$ whenever it holds that

$$p\leq\frac{1}{R_{\mathcal{X}}(\psi_2)}\frac{G_{\mathcal{X}}(\psi_1)}{1-G_{\mathcal{X}}(\psi_1)}. \quad (4)$$

Since it can be guaranteed that $R_{\mathcal{X}}(\psi_2)>0$ and $0<G_{\mathcal{X}}(\psi_1)<1$ when $\psi_1,\psi_2\notin\mathcal{X}$, there always exists $p\in(0,1]$ such that Eq. (6) holds. Theorem 2 then arises as a corollary as, given any $\psi_1\notin\mathcal{X}$, continuity arguments show that there always exists an inequivalent $\psi_2\notin\mathcal{X}$ with $R_{\mathcal{X}}(\psi_2)$ small enough so that one can take $p=1$ in Eq. (6) and construct a CPTP map.

Theorem 1 proves that in our case there are no inequivalent forms of entanglement. This is in sharp contrast to LOCC where, leaving aside the case $H=(\mathbb{C}^2)^{\otimes 3}$, the state space splits into a cumbersome zoology of infinitely many different SLOCC classes of unrelated entangled states. Theorem 2 provides the non-triviality of our theories. While almost all states turn out to be isolated under LOCC [18], our classes of free operations induce a meaningful partial order structure where, as in the case of bipartite entanglement, every pure state can be transformed into a more weakly entangled pure state. It is important to mention that the result of [18] proves generic isolation when transformations are restricted among GME states with the rank of all n single-particle reduced density matrices equal to d. However, Theorem 2 still holds under this restriction [31].

Existence of a maximally resourceful state. Theorems 1 and 2 show that limitations of the resource theory of multipartite entanglement under LOCC can be overcome if one considers FSP or BSP operations instead. These positive results raise the question of whether the induced structure is powerful enough to have a unique multipartite maximally entangled state. If this were so, our theories would point to a relevant class of states that should be at the heart of the applications of multipartite entanglement in a similar fashion to the maximally entangled state in the bipartite case. In order to answer this question, we provide first an unambiguous definition of a maximally resourceful state which, on the analogy of the bipartite case, depends on the number of parties n and local dimension d: a state ψ on H is the maximally resourceful state on H if it can be transformed by means of the free operations into any other state on H [**?**]. We analyze first the case of FSP operations, where we find a negative answer to the above question.

Theorem 3. *In the resource theory of entanglement where the free operations are FSP maps, there exists no maximally entangled state on $H = (\mathbb{C}^2)^{\otimes 3}$.*

Although the details of the proof are given in [31], we outline here its structure. First, we use that if a maximally entangled state in this case existed, it would need to be the W state $|W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$. This is because it has been shown in [32] that the W state is the unique state in this Hilbert space that achieves the maximal possible value of $G_{\mathcal{FS}}$, which we have shown above to be an FSP-measure. Thus, if there existed a maximally entangled state, it would be necessary that the W state could be transformed by FSP into any other state. However, we show that there exists no FSP map transforming the W state into the GHZ state ($|GHZ(3,2)\rangle$ in Eq. (5) below). To verify this last claim, it suffices to find an FSP-measure E such that $E(GHZ) > E(W)$. However, as discussed above, not many FSP-measures are known and, as with the geometric measure, it is also known that the relative entropy of entanglement of the W is larger than that of the GHZ state [33]. This leaves us then with the robustness measure $R_{\mathcal{FS}}$, for which we are able to show that $R_{\mathcal{FS}}(W) = R_{\mathcal{FS}}(GHZ) = 2$. This alone does not forbid that $W \to_{FSP} GHZ$ but from the insight developed in computing these quantities, an obstruction to such transformation can be found even though they are equally robust. It is worth mentioning that, to our knowledge, this is the first time that the robustness is computed for multipartite states and we have reasons to conjecture that the W and GHZ states attain its maximal value on H, being the only states that do so.

Theorem 3 forbids then the existence of a multipartite maximally entangled state under FSP in the simplest case of $H = (\mathbb{C}^2)^{\otimes 3}$. However, it is instructive to compare with the LOCC scenario since these values of n and d make up the only case where no state is isolated in the latter formalism (aside from the bipartite case). We show in [31] that the W and GHZ states can be transformed by FSP operations into states that are not obtainable from any other 3-qubit states by LOCC. These states might be chosen to lie in different SLOCC classes, so, additionally, this provides an explicit example of deterministic FSP conversions among states in different SLOCC classes.

Finally, we study the resource theory under BSP operations where, remarkably, we find a unique maximally GME state for any value of n and d, given by the generalized GHZ state

$$|GHZ(n,d)\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} |i\rangle^{\otimes n}. \tag{5}$$

Theorem 4. *In the resource theory of entanglement where the free operations are BSP maps, there exists a maximally GME state on every H. Namely, $\forall|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$, there exists a CPTP BSP map Λ such that $\Lambda(GHZ(n,d)) = \psi$.*

The complete proof of this result is given in [31]. The main idea behind it is to use again the construction of the proof of Theorems 1 and 2, that shows that there is a CPTP BSP map Λ such that $\Lambda(GHZ(n,d)) = \psi$ if $R_{\mathcal{BS}}(\psi) \leq G_{\mathcal{BS}}(GHZ(n,d))/(1 - G_{\mathcal{BS}}(GHZ(n,d)))$ (cf. Eq. (6)). However, unlike for the FS case, $G_{\mathcal{BS}}$ is straightforward to compute [34] in terms of the Schmidt decomposition across every possible bipartite splitting of the parties $M|\bar{M}$ ($|\psi\rangle = \sum_i \sqrt{\lambda_i^{M|\bar{M}}}|i\rangle_M|i\rangle_{\bar{M}}$) as

$$G_{\mathcal{BS}}(\psi) = 1 - \max_{M \subsetneq [n]} \lambda_1^{M|\bar{M}}, \tag{6}$$

where $\lambda_1^{M|\bar{M}}$ is the largest Schmidt coefficient of ψ in the corresponding splitting. This immediately shows that the generalized GHZ state has maximal value of the geometric measure, $G_{\mathcal{BS}}(GHZ(n,d)) = (d-1)/d$. Finally, a simple estimate shows that $R_{\mathcal{BS}}(\psi) \leq d - 1$ $\forall|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$, which leads to the desired result.

It follows from the proof that it suffices to have maximal $G_{\mathcal{BS}}$ to be convertible to any other state by BSP operations. Thus, any state fulfilling that $G_{\mathcal{BS}} = (d-1)/d$ must automatically maximize any other BSP-measure. More importantly, this also shows that any two states achieving this value of the geometric measure are deterministically interconvertible by BSP operations and, therefore, belong to the same GME-equivalence class despite potentially not being related by local unitary transformations. An example of such class when $d = 2$ are GME graph states for which it is known that $G_{\mathcal{BS}} = 1/2$ [35]. Hence, all graph states including the generalized GHZ state are in the equivalence class of the maximally GME state in this theory. It is remarkable to find that this very relevant family of states [36] in quantum computation and error correction has this feature in a resource theory of GME and we believe this is worth further research. Another previously considered family of states that belongs to this equivalence class is that of absolutely maximally entangled (AME) states [37], which is defined as those states for which all reduced density matrices are proportional to the identity in the maximum possible dimensions. It follows from Eq. (6) that for all AME states it holds that $G_{\mathcal{BS}} = (d-1)/d$ (for those values of n and d for which they exist). Equation (6) also tells us that a necessary condition for a state to be in the equivalence class of the maximally GME state is that all single-particle reduced density matrices must be proportional to the d-dimensional identity. However, this condition is not sufficient: the state in $(\mathbb{C}^2)^{\otimes 4}$ $|\phi\rangle = \sqrt{p}|\phi^+\rangle_{12}|\phi^+\rangle_{34} + \sqrt{1-p}|\phi^-\rangle_{12}|\phi^-\rangle_{34}$ ($|\phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$) is a GME state (if $p \neq 0, 1$) with this property but $G_{\mathcal{BS}}(\phi) < 1/2$ (if $p \neq 1/2$).

Conclusions. We have shown that non-trivial (i.e. without isolation) resource theories of multipartite entanglement are possible in which moreover inequivalent forms of entanglement do not exist. However, no resource theory of non-full-separability can have a maximally entangled state for 3-qubit states since this is not possible under FSP transformations, the largest conceivable class of free operations (future work should study whether this no-go result generalizes to other values of n and d). On the other hand, the BSP paradigm induces a resource theory of GME with a maximally resourceful state. Given this positive result, it would be interesting to analyze further features of this theory and, in particular, whether an operational grounding to this conceptually satisfying structure can be found. We also note that GME does not fulfill Axiom 1 of [21], so it is open whether an asymptotically reversible theory of this resource is possible.

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[47] Notice that this already implies that there exists no free operation that transforms an inequivalent state on H into ψ.

1

Supplemental material

We prove the theorems introduced in the main text. For the reader's convenience, we provide the necessary definitions and restate the theorems.

Throughout the proof we will use repeatedly that, if ρ_1 and ρ_2 are density matrices, the map

$$\Lambda(\rho) = \operatorname{tr}(A\rho)\rho_1 + \operatorname{tr}[(\mathbb{1} - A)\rho]\rho_2 \tag{1}$$

is CPTP if $0 \leq A \leq \mathbb{1}$ (see e.g. [1]).

We say that a functional E taking operators on H to non-negative real numbers is an FSP-measure (BSP-measure) if $E(\rho) \geq E(\Lambda(\rho))$ for every state ρ and FSP (BSP) map Λ. Measures of the form

$$E_{\mathcal{X}}(\rho) = \inf_{\sigma \in \mathcal{X}} E(\rho||\sigma), \tag{2}$$

where $\mathcal{X}$ stands for either $\mathcal{FS}$ or $\mathcal{BS}$, have the corresponding monotonicity property as long as the distinguishability measure $E(\rho||\sigma)$ is contractive, i.e. $E(\Lambda(\rho)||\Lambda(\sigma)) \leq E(\rho||\sigma)$ for every $\rho, \sigma \in H$ and every CPTP map Λ.

As already explained in the main text, two FSP- (BSP-)measures that will play a key role in developing the resource theory of FSP (BSP) operations are the geometric measure and the robustness. For the reader's convenience, we recall their definitions. The robustness is given by

$$R_{\mathcal{X}}(\cdot) = \min_{\sigma \in \mathcal{X}} R(\cdot||\sigma) \tag{3}$$

where

$$R(\rho||\sigma) = \min\left\{ s : \frac{\rho + s\sigma}{1+s} \in \mathcal{X} \right\}, \tag{4}$$

and the geometric measure, which we only need to consider here for pure states, boils down to

$$G_{\mathcal{X}}(\cdot) = 1 - \left(\max_{|\phi\rangle \in \mathcal{X}} |\langle \phi|\cdot\rangle| \right)^2. \tag{5}$$

I. NON-TRIVIALITY OF THE THEORIES

Theorem 1. *In a resource theory of entanglement where the free operations are $\mathcal{X}$-preserving maps, all resource states are interconvertible with non-zero probability, i.e. given any pure $\psi_1, \psi_2 \notin \mathcal{X}$, there exists a completely positive and trace non-increasing $\mathcal{X}$-preserving map Λ such that $\Lambda(\psi_1) = p\psi_2$ with $p \in (0, 1]$.*

Proof. Notice that, since $\psi_1, \psi_2 \notin \mathcal{X}$ and both the geometric measure and the robustness are faithful measures [2, 3], $R_{\mathcal{X}}(\psi_2), G_{\mathcal{X}}(\psi_1) > 0$. Also, $G_{\mathcal{X}}(\psi_1) < 1$ because the fully (bi-)separable states span the whole Hilbert space. Pick $p \in (0, 1]$ such that

$$p \leq \frac{1}{R_{\mathcal{X}}(\psi_2)} \frac{G_{\mathcal{X}}(\psi_1)}{1 - G_{\mathcal{X}}(\psi_1)} \tag{6}$$

and let

$$\Lambda(\eta) = p \operatorname{tr}(\psi_1 \eta)\psi_2 + \operatorname{tr}\left[(\mathbb{1} - \psi_1)\eta\right] \rho_{\mathcal{X}}. \tag{7}$$

Here $\rho_{\mathcal{X}} \in \mathcal{X}$ is the state which gives the corresponding robustness of ψ_2, i.e., $R_{\mathcal{X}}(\psi_2) = R(\psi_2||\rho_{\mathcal{X}})$—cf. equation (4). (Note that Λ can be completed to a CPTP $\mathcal{X}$-preserving map by adding a term of the form $\Lambda'(\eta) = (1-p)\operatorname{tr}(\psi_1\eta)\rho_{\mathcal{X}}$.) Then $\Lambda(\psi_1) = p\psi_2$ and it remains to be shown that Λ is $\mathcal{X}$-preserving. Let $\sigma \in \mathcal{X}$. Then

$$\Lambda(\sigma) \propto \psi_2 + \frac{1}{p}\left(\frac{1}{\operatorname{tr}(\psi_1\sigma)} - 1 \right) \rho_{\mathcal{X}}, \tag{8}$$

so $\Lambda(\sigma)/\operatorname{tr}(\Lambda(\sigma)) \in \mathcal{X}$ iff $\frac{1}{p}\left(\frac{1}{\operatorname{tr}(\psi_1\sigma)} - 1\right) \geq R_{\mathcal{X}}(\psi_2)$. But this holds from equation (6) and using $\operatorname{tr}(\psi_1\sigma) \leq 1 - G_{\mathcal{X}}(\psi_1)\ \forall \sigma \in \mathcal{X}$. □

Theorem 2. *In a resource theory of entanglement where the free operations are $\mathcal{X}$-preserving maps, no resource state is isolated, i.e. given any pure $\psi_1 \notin \mathcal{X}$ on H, there exists an inequivalent pure $\psi_2 \notin \mathcal{X}$ on H and a CPTP $\mathcal{X}$-preserving map Λ such that $\Lambda(\psi_1) = \psi_2$.*

Proof. Consider the map (7) from the proof of Theorem 1. This map can be made deterministic if $R_{\mathcal{X}}(\psi_2)$ is sufficiently smaller than $G_{\mathcal{X}}(\psi_1)$. Indeed, if

$$\frac{1}{R_{\mathcal{X}}(\psi_2)} \frac{G_{\mathcal{X}}(\psi_1)}{1 - G_{\mathcal{X}}(\psi_1)} > 1, \tag{9}$$

then we can pick $p = 1$ in the map (7) so Λ is CPTP (see equation (1)). Since robustness is a continuous function of the input state [3], it can be arbitrarily close to zero and so there exists ψ_2 such that the above condition is fulfilled for any ψ_1. Further, ψ_1, ψ_2 are inequivalent if they have different robustness, but $R(\psi_2)$ can always be picked to be different from $R(\psi_1)$ and still satisfying equation (9). □

The generic isolation result proven in Ref. [4] holds when transformations are restricted among GME states fully supported on H (i.e. such that all n single-particle reduced density matrices have rank d). Importantly, ψ_1 and ψ_2 in Theorem 2 may both be fully supported on H, in contrast to the LOCC scenario. It suffices to consider

$$\begin{aligned} |\psi_2\rangle = \sqrt{1-\varepsilon}\,|0\rangle^{\otimes n} + \sqrt{\frac{\varepsilon}{d-1}}\,|1\rangle^{\otimes n} \\ + \cdots + \sqrt{\frac{\varepsilon}{d-1}}\,|d-1\rangle^{\otimes n} \end{aligned} \tag{10}$$

as an example of a GME fully supported state on H which meets the requirements for small enough ε.

II. FSP REGIME

Theorem 3. *In the resource theory of entanglement where the free operations are FSP maps, there exists no maximally entangled state on* $H = (\mathbb{C}^2)^{\otimes 3}$.

To prove this theorem, it is useful to introduce the following two lemmas in order to compute the robustness of the W and GHZ states.

Lemma 1. $R_{\mathcal{FS}}(GHZ) = 2$.

Proof. The robustness can be bounded from above from the definition (equations (3), (4)), as any fully separable state which is a convex combination of the GHZ state with a fully separable state will give an upper bound to the robustness. Ref. [5] provides a dual characterization in terms of entanglement witnesses which we use to bound the robustness from below:

$$R_{\mathcal{FS}}(\rho) = \max\left\{0, -\min_{\mathcal{W}\in\mathcal{M}} \mathrm{tr}(\mathcal{W}\rho)\right\}. \tag{11}$$

A witness for a state ρ is an operator $\mathcal{W}$ such that $\mathrm{tr}(\mathcal{W}\sigma) \geq 0$ for all $\sigma \in \mathcal{FS}$ and $\mathrm{tr}(\mathcal{W}\rho) < 0$. If the witness also satisfies $\mathrm{tr}(\mathcal{W}\sigma) \leq 1$ for all $\sigma \in \mathcal{FS}$ (which defines the set $\mathcal{M}$ above), then $-\mathrm{tr}(\mathcal{W}\rho)$ is a lower bound to the robustness.

First, we show $R_{\mathcal{FS}}(GHZ) \leq 2$. We will use the following notation as a means to characterize full separability of certain states (this is a simplified version of the separability criterion in [6, §2.1]): a state of the form

$$\begin{aligned} &\rho(\lambda^+, \lambda^-, \lambda) = \\ &\quad \lambda^+ GHZ + \lambda^- GHZ_- + \frac{\lambda}{6}\sum_{i=001}^{110} |i\rangle\langle i|, \end{aligned} \tag{12}$$

where $|GHZ_-\rangle = (|000\rangle - |111\rangle)/\sqrt{2}$ and the summation index i ranges from 001 to 110 in binary, is fully separable iff

$$|\lambda^+ - \lambda^-| \leq \lambda/3. \tag{13}$$

We must also have $\lambda^+ + \lambda^- + \lambda = 1$ for normalization, and $\lambda^\pm, \lambda \geq 0$ for $\rho(\lambda^+, \lambda^-, \lambda)$ to be positive. Thus, the set of fully separable states of the form (12) is a polytope, and this property will be used later.

Consider the following state:

$$\frac{1}{3}\left(GHZ + 2\rho\left(0, \frac{1}{4}, \frac{3}{4}\right)\right) = \rho\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right), \tag{14}$$

It is straightforward to check that both $\rho\left(0, \frac{1}{4}, \frac{3}{4}\right)$ and $\rho\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right)$ satisfy (13) with equality, so $R_{\mathcal{FS}}(GHZ) \leq 2$.

Next, we show $R_{\mathcal{FS}}(GHZ) \geq 2$. Let

$$\mathcal{W} = \frac{2}{3}\mathbb{1} - \frac{8}{3}GHZ + \frac{4}{3}GHZ_- \tag{15}$$

be a candidate witness for this purpose. To show $0 \leq \mathrm{tr}(\mathcal{W}\sigma) \leq 1$ for all fully separable states σ, it is enough to restrict to states σ of the form (12), as can be shown by considering the twirling map T_{GHZ} onto the GHZ-symmetric subspace. This map is defined in [7], but we will only need the following properties: it is FSP and self-dual, it maps all states onto states of the form (12), i.e.

$$T_{GHZ}(\tau) = \rho(\lambda^+, \lambda^-, \lambda) \tag{16}$$

for every state τ on H and for some $\lambda^\pm, \lambda$ and, moreover, these states are fixed points: $T_{GHZ}(\rho(\lambda^+, \lambda^-, \lambda)) = \rho(\lambda^+, \lambda^-, \lambda)$ for all $\lambda^\pm, \lambda$. In particular, $T_{GHZ}(GHZ) = GHZ$ and the witness $\mathcal{W}$ in equation (15) is such that $T_{GHZ}(\mathcal{W}) = \mathcal{W}$, and so

$$\mathrm{tr}(\mathcal{W}\sigma) = \mathrm{tr}(T_{GHZ}(\mathcal{W})\,\sigma) = \mathrm{tr}(\mathcal{W}\,T_{GHZ}(\sigma)) \tag{17}$$

holds for any state σ. Therefore, if $0 \leq \mathrm{tr}(\mathcal{W}\sigma) \leq 1$ holds for all $\sigma \in \mathcal{FS}$ such that $T_{GHZ}(\sigma) = \sigma$, i.e. those of the form (12) where (13) holds [8, 9], then it is guaranteed to hold for any $\sigma \in \mathcal{FS}$.

As the space of fully separable GHZ-symmetric states is a polytope, it is enough to show that $0 \leq \mathrm{tr}(\mathcal{W}\sigma) \leq 1$ at the vertices of the polytope, which are (cf. [8, 9]):

$$\begin{aligned} \sigma_1 &= \rho(0, 0, 1) \\ \sigma_2 &= \rho\left(0, \frac{1}{4}, \frac{3}{4}\right) \\ \sigma_3 &= \rho\left(\frac{1}{2}, \frac{1}{2}, 0\right) \\ \sigma_4 &= \rho\left(\frac{1}{4}, 0, \frac{3}{4}\right). \end{aligned} \tag{18}$$

It is straightforward to check that $0 \leq \mathrm{tr}(\mathcal{W}\sigma_j) \leq 1$ for all $j = 1, ..., 4$. Since $\mathrm{tr}(\mathcal{W}\,GHZ) = -2 < 0$, $\mathcal{W}$ is a witness for the GHZ-state that meets the required condition and so $R_{\mathcal{FS}}(GHZ) \geq 2$. □

Lemma 2. $R_{\mathcal{FS}}(W) = 2$.

Proof. The strategy is similar to the proof of Lemma 1. First, we prove $R_{\mathcal{FS}}(W) \leq 2$. We will show that

$$\eta = \frac{1}{3}(W + 2\tau), \tag{19}$$

where

$$\eta = \frac{9}{16}|000\rangle\langle 000| + \frac{3}{16}|111\rangle\langle 111| + \frac{1}{16}W + \frac{3}{16}\overline{W} \tag{20}$$

and

$$\tau = \frac{3}{8}\left|000\right\rangle\!\left\langle 000\right| + \frac{1}{8}\left|111\right\rangle\!\left\langle 111\right| + \frac{3}{8}W + \frac{1}{8}\overline{W} \qquad (21)$$

are both fully separable. Here and in what follows, $\overline{W}$ denotes the qubit-flipped version of the W-state,

$$\left|\overline{W}\right\rangle = \frac{1}{\sqrt{3}}\left(|110\rangle + |101\rangle + |011\rangle\right) . \qquad (22)$$

As shown in Theorem 6.2 of [10], if a symmetric 3-qubit state remains positive after partial transposition (PPT), then it is FS. Since both η and τ are symmetric 3-qubit states, it is enough to check that they are PPT, which is readily done, to conclude that they are fully separable.

Another way to see this is by writing η and τ as a convex combination of fully separable states using a result from [11]. Observe that

$$\begin{aligned}\eta =& \frac{5}{9}\left|000\right\rangle\!\left\langle 000\right| \\ &+\frac{4}{9}\left(\frac{1}{2^6}\left|000\right\rangle\!\left\langle 000\right| + \frac{27}{2^6}\left|111\right\rangle\!\left\langle 111\right|\right. \\ &\left.+\frac{9}{2^6}W + \frac{27}{2^6}\overline{W}\right)\end{aligned} \qquad (23)$$

and

$$\begin{aligned}\tau =& \frac{1}{9}\left|111\right\rangle\!\left\langle 111\right| \\ &+\frac{8}{9}\left(\frac{27}{2^6}\left|000\right\rangle\!\left\langle 000\right| + \frac{1}{2^6}\left|111\right\rangle\!\left\langle 111\right|\right. \\ &\left.+\frac{27}{2^6}W + \frac{9}{2^6}\overline{W}\right)\end{aligned} \qquad (24)$$

where, in each case, the first term is clearly fully separable. As we shall see, the second term is of the form

$$\begin{aligned}&\mathrm{tr}(\phi^{\otimes 3}\left|000\right\rangle\!\left\langle 000\right|)\left|000\right\rangle\!\left\langle 000\right| \\ &+\mathrm{tr}\big(\phi^{\otimes 3}\left|111\right\rangle\!\left\langle 111\right|\big)\left|111\right\rangle\!\left\langle 111\right| \\ &+\mathrm{tr}\big(\phi^{\otimes 3}\,W\big)\,W \\ &+\mathrm{tr}\big(\phi^{\otimes 3}\,\overline{W}\big)\,\overline{W}\end{aligned} \qquad (25)$$

for some qubit state ϕ. Ref. [11] shows that all states of this form are fully separable. Writing

$$|\phi\rangle = \cos\alpha\,|0\rangle + \mathrm{e}^{\mathrm{i}\beta}\sin\alpha\,|1\rangle . \qquad (26)$$

and inserting it into equation (25), the parameter β cancels in all terms and the state in equation (25) can be written in terms of α alone with $\alpha = \pi/3$ for η and $\alpha = \pi/6$ for τ.

Next, we prove $R_{\mathcal{FS}}(W) \geq 2$. We will show that

$$\begin{aligned}A =& \left|000\right\rangle\!\left\langle 000\right| - 3\,W + \\ &\left|001\right\rangle\!\left\langle 001\right| + \left|010\right\rangle\!\left\langle 010\right| + \left|100\right\rangle\!\left\langle 100\right| + 3\,\overline{W}\end{aligned} \qquad (27)$$

is a witness for the state $\left|W\right\rangle\!\left\langle W\right|$ such that

$$\mathrm{tr}(AW) = -2 \qquad (28)$$

and

$$0 \leq \mathrm{tr}(A\sigma) \leq 1 \qquad (29)$$

for all $\sigma \in \mathcal{FS}$.

Let $\sigma \in \mathcal{FS}$. Without loss of generality, to prove (29) we can assume $\sigma = \left|\psi\right\rangle\!\left\langle\psi\right|$ is pure. So we want to show

$$0 \leq \mathrm{tr}(A\left|\psi\right\rangle\!\left\langle\psi\right|) \leq 1 . \qquad (30)$$

Notice that A is permutationally invariant, and that we can express A in the basis of Pauli matrices as

$$A = \sum_{ijk\in x,y,z} \lambda_{ijk}\sigma_i \otimes \sigma_j \otimes \sigma_k + \frac{\mathbb{1}_8}{2} \qquad (31)$$

for some $\lambda_{ijk} \in \mathbb{R}$ and where $\mathbb{1}_d$ is the d-dimensional identity, so that

$$A' = A - \frac{\mathbb{1}_8}{2} \qquad (32)$$

has no identity component in the basis of Pauli matrices. That is, A' contains only full correlation terms, and it is still permutationally invariant so it satisfies the conditions of Corollary 5 (ii) in [12]. In particular, A' can be viewed as a symmetric three-linear form acting on $\mathbb{R}^3$. This means that

$$\max_{|\psi\rangle\in\mathcal{FS}} \left|\mathrm{tr}(A'\left|\psi\right\rangle\!\left\langle\psi\right|)\right| \qquad (33)$$

can be attained by a symmetric state $|\psi\rangle = |a\rangle\,|a\rangle\,|a\rangle \equiv |aaa\rangle$. The qubit $|a\rangle$ can be expressed in terms of two real parameters as

$$|a\rangle = \cos\alpha\,|0\rangle + \mathrm{e}^{\mathrm{i}\beta}\sin\alpha\,|1\rangle \qquad (34)$$

and so

$$\left|\mathrm{tr}(A'\left|aaa\right\rangle\!\left\langle aaa\right|)\right| = \frac{1}{2}\left|\cos 6\alpha\right| \leq \frac{1}{2} . \qquad (35)$$

But this completes the proof, since, by linearity, to show

$$-\frac{1}{2} \leq \mathrm{tr}(A'\left|\psi\right\rangle\!\left\langle\psi\right|) \leq \frac{1}{2} \qquad (36)$$

(which is equivalent to (30)) it suffices to show

$$\max_{|\psi\rangle\in\mathcal{FS}} \left|\mathrm{tr}(A'\left|\psi\right\rangle\!\left\langle\psi\right|)\right| \leq \frac{1}{2} . \qquad (37)$$

This can be seen by viewing $\mathrm{tr}(A'\left|\psi\right\rangle\!\left\langle\psi\right|)$ as a symmetric three-linear form in $\mathbb{R}^3$. If the maximum absolute value is attained by some state $|a^*\rangle$, then the state $|\tilde{a}^*\rangle$ which flips the sign of the vector which the three-linear form acts on will give a minimum of the expression equal to minus the maximum. Hence,

$$\begin{aligned}\max_{|\psi\rangle\in\mathcal{FS}} |\mathrm{tr}(A'\,|\psi\rangle\langle\psi|)| &= \max_{|\psi\rangle\in\mathcal{FS}} \mathrm{tr}(A'\,|\psi\rangle\langle\psi|) \\ &= -\min_{|\psi\rangle\in\mathcal{FS}} \mathrm{tr}(A'\,|\psi\rangle\langle\psi|)\,.\end{aligned} \tag{38}$$

Therefore (30) holds true and hence the witness A gives the stated lower bound for the FS robustness of W. □

We note that the values obtained for the robustness $R_{\mathcal{FS}}$ of the W and GHZ states show that, unlike in the bipartite case, the robustness can be strictly larger than the generalized robustness. The generalized robustness, $R_G(\cdot)$, is defined as

$$R_G(\cdot) = \min_{\tau\in H} R(\cdot||\tau) \tag{39}$$

where, this time, τ may be separable or entangled. Hence $R_G(\cdot) \leq R(\cdot)$ but, in addition, it was shown in [13] that $R_G(\cdot) = R(\cdot)$ for bipartite pure states. However, the generalized robustness of the W state has been computed in [11] to be 5/4, and that of the GHZ state was shown to be 1 in [7], so they are both strictly less than the robustness of these states. To the best of our knowledge, this is the first time that states such that $R_G(\cdot) < R(\cdot)$ have been found.

We are now ready to prove Theorem 3.

Proof. As we outlined in the main text, the only candidate for a maximally entangled state of three qubits is the W state, as it is the unique state on $H = \left(\mathbb{C}^2\right)^{\otimes 3}$ that achieves the maximum value of the FSP-measure $G_{\mathcal{FS}}$ (among both pure and mixed states, since the convex-roof extension of $G_{\mathcal{FS}}$ to mixed states ensures that the maximum value will always be achieved by a pure state). So, if there existed a maximally entangled state, it would need to be possible that the W state be transformed into any other state via an FSP map. We will assume that there exists an FSP map Λ such that $\Lambda(W) = GHZ$, and will arrive at a contradiction by showing that there exists a state $\eta \in \mathcal{FS}$ such that $\Lambda(\eta) \notin \mathcal{FS}$.

Let Λ be an FSP map such that $\Lambda(W) = GHZ$ and let

$$\eta = \frac{1}{3}W + \frac{2}{3}\tau \in \mathcal{FS}\,, \tag{40}$$

where $\tau, \eta \in \mathcal{FS}$, be the convex combination that gives the upper bound to $R_{\mathcal{FS}}(W)$ in equations ((19)-(21)). Let T_{GHZ} be the twirling map onto the GHZ-symmetric subspace (defined in [11]; see also the proof of Lemma 1). Then,

$$\begin{aligned}\eta' =& T_{GHZ}\left(\Lambda\left(\frac{1}{3}W + \frac{2}{3}\tau\right)\right) \\ =& \frac{1}{3}GHZ + \frac{2}{3}T_{GHZ}(\Lambda(\tau))\,.\end{aligned} \tag{41}$$

Since both T_{GHZ} and Λ are full separability-preserving, it is the case that $\eta', \Lambda(\tau)$, $T_{GHZ}(\Lambda(\tau)) \in \mathcal{FS}$. Now, recall that τ has a non-zero W component:

$$\tau = p\,|W\rangle\langle W| + (1-p)\xi$$

for some $p \in (0,1)$ and some state ξ, so that

$$\eta' = \frac{1}{3}GHZ + \frac{2}{3}\left[p\,GHZ + (1-p)T_{GHZ}(\Lambda(\xi))\right]\,. \tag{42}$$

But, as we shall now show, the FS GHZ-symmetric state υ such that

$$\frac{1}{3}GHZ + \frac{2}{3}\upsilon \in \mathcal{FS} \tag{43}$$

is unique, i.e. if equation (43) holds then necessarily $\upsilon = \rho(0, 1/4, 3/4)$ as in equation (14). However, the state appearing in equation (42) is not υ (since $\mathrm{tr}(\upsilon\,GHZ) = 0$) hence, contrary to our assumption, η' cannot be FS.

Recall, from the proof of Lemma 1 (equation (12)), that all GHZ-symmetric states are of the form

$$\begin{aligned}&\rho(\lambda^+, \lambda^-, \lambda) = \\ &\qquad \lambda^+ GHZ + \lambda^- GHZ_- + \frac{\lambda}{6}\sum_{i=001}^{110} |i\rangle\langle i|\end{aligned} \tag{44}$$

so that equation (43) can be expressed in terms of the λ parameters as

$$\frac{1}{3}GHZ + \frac{2}{3}\rho(\lambda^+, \lambda^-, \lambda) = \rho(\frac{1}{3} + \frac{2}{3}\lambda^+, \frac{2}{3}\lambda^-, \frac{2}{3}\lambda)\,. \tag{45}$$

States of the form (44) are fully separable iff

$$|\lambda^+ - \lambda^-| \leq \lambda/3\,. \tag{46}$$

Since this condition must hold for both states $\rho(\cdot,\cdot,\cdot)$ in equation (45), we must also have

$$\left|\frac{1}{3} + \frac{2}{3}\lambda^+ - \frac{2}{3}\lambda^-\right| \leq \frac{2}{9}\lambda \tag{47}$$

and, for normalisation, we need

$$\lambda^+ + \lambda^- + \lambda = 1\,. \tag{48}$$

It is straightforward to check that these three conditions hold only if

$$\lambda^+ = 0;\ \lambda^- = 1/4;\ \lambda = 3/4\,, \tag{49}$$

which corresponds to the state υ as claimed above.

Therefore η in equation (40) is fully separable, yet $\eta' = T_{GHZ}(\Lambda(\eta))$ is not fully separable. So Λ is not FSP and hence the theorem is proven. □

Thus, Theorem 3 forbids the existence of a multipartite maximally entangled state under FSP in the simplest case of $H = (\mathbb{C}^2)^{\otimes 3}$. Still, the LOCC case provides a useful comparison since, in this formalism, the case $n = 3$, $d = 2$ is the only one in which no state is isolated (in addition to $n = 2$). Whenever no single maximally entangled state exists one needs to consider a maximally entangled set (MES) [14], defined as the minimal set of states on H such that any state on H can be obtained by means of the free operations from a state in this set. The MES under LOCC for $n = 3$ and $d = 2$ has been characterized in [14], and it is found to be relatively small in the sense that it has measure zero on H (in contrast, for other values of n and d the fact that isolation is generic imposes that the MES has full measure on H). However, interestingly, the MES under FSP is smaller even in this case, given that it is strictly included in the MES under LOCC. This is because, as we will now show, the W and GHZ states can be transformed by FSP operations into inequivalent states that are in the MES under LOCC. It is worth mentioning that the target states may be chosen to lie in different SLOCC classes with respect to the initial states, and so this gives an explicit example of deterministic FSP conversions among states in different SLOCC classes.

Let ψ^+_{GHZ} denote states of the form

$$|\psi^+_{GHZ}\rangle = \sqrt{K}\left(|000\rangle + |\phi_A\phi_B\phi_C\rangle\right) \tag{50}$$

where

$$\begin{aligned} |\phi_A\rangle &= \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle\ , \\ |\phi_B\rangle &= \cos\beta\,|0\rangle + \sin\beta\,|1\rangle\ , \\ |\phi_C\rangle &= \cos\gamma\,|0\rangle + \sin\gamma\,|1\rangle\ , \end{aligned} \tag{51}$$

$\alpha, \beta, \gamma \in (0, \pi/2]$ and $K = (2(1+\cos\alpha\cos\beta\cos\gamma))^{-1}$ is a normalisation factor. States of the form ψ^+_{GHZ} are in the MES under LOCC, since they cannot be reached by any LOCC map regardless of the input state on $H = (\mathbb{C}^2)^{\otimes 3}$ [14–16]. So the following proposition does not hold in the LOCC regime.

Proposition 1. *There exists an FSP map Λ such that $\Lambda(W) = \psi^+_{GHZ}$ for some state of the form ψ^+_{GHZ}.*

Proof. Let

$$\Lambda(\eta) = \mathrm{tr}(W\eta)\psi^+_{GHZ} + \mathrm{tr}\left[(\mathbb{1} - W)\eta\right]\tau_{\mathcal{FS}}, \tag{52}$$

where $\tau_{\mathcal{FS}} \in \mathcal{FS}$ is the state that gives the robustness of the state ψ^+_{GHZ}. Clearly, $\Lambda(W) = \psi^+_{GHZ}$ and it remains to be shown that Λ is FSP. As argued in Theorems 1 and 2, this happens when

$$R_{\mathcal{FS}}(\psi^+_{GHZ}) \leq \frac{G_{\mathcal{FS}}(W)}{1 - G_{\mathcal{FS}}(W)} = \frac{5}{4}\,. \tag{53}$$

But, by continuity of the robustness, such a state ψ^+_{GHZ} can always be found by picking the parameters α, β, γ sufficiently close to zero since in this case the states ψ^+_{GHZ} approach the set of FS states.

Anyway, for the sake of completeness, we provide an explicit quantitative upper bound in what follows. Consider the invertible local operations

$$\begin{aligned} A &= \begin{pmatrix} 1 & \cos\alpha \\ 0 & \sin\alpha \end{pmatrix}, \\ B &= \begin{pmatrix} 1 & \cos\beta \\ 0 & \sin\beta \end{pmatrix}, \\ C &= \begin{pmatrix} 1 & \cos\gamma \\ 0 & \sin\gamma \end{pmatrix}. \end{aligned} \tag{54}$$

Applying these to the FS states in equation (14) used to bound the robustness of the GHZ state,

$$A \otimes B \otimes C\left(\frac{1}{3}GHZ + \frac{2}{3}\upsilon\right)A^\dagger \otimes B^\dagger \otimes C^\dagger\,, \tag{55}$$

gives a state proportional to

$$\frac{1}{3}\left(1 + \cos\alpha\cos\beta\cos\gamma\right)\psi^+_{GHZ} + \frac{2}{3}\frac{4 - \cos\alpha\cos\beta\cos\gamma}{4}\upsilon'\,, \tag{56}$$

where $\upsilon' = A \otimes B \otimes C\upsilon A^\dagger \otimes B^\dagger \otimes C^\dagger$ is still fully separable since local operations cannot create entanglement. For the same reason, the state in equation (56) is fully separable, and hence the robustness of the state ψ^+_{GHZ} cannot exceed [**?**]

$$R_{\mathcal{FS}}(\psi^+_{GHZ}) \leq \frac{4 - \cos\alpha\cos\beta\cos\gamma}{2(1 + \cos\alpha\cos\beta\cos\gamma)}\,. \tag{57}$$

Clearly, there exist $\alpha, \beta, \gamma \in (0, \pi/2]$ such that this bound is lower than or equal to $5/4$, as required. For an example, take $\alpha = \beta = \pi/2$ and γ such that $\cos\gamma \geq 6/7$. □

We will now show the converse result: there are FSP maps which take the GHZ-state to states in the W-class which are in the MES under LOCC. Such states are of the form

$$|\psi_W\rangle = \sqrt{x_1}\,|001\rangle + \sqrt{x_2}\,|010\rangle + \sqrt{x_3}\,|100\rangle \tag{58}$$

where $x_1 + x_2 + x_3 = 1$. They are in the MES under LOCC, as no LOCC map can reach these states for any input state on $H = (\mathbb{C}^2)^{\otimes 3}$ [14, 15, 17], but (as we will now prove) not under FSP.

Proposition 2. *There exists an FSP map Λ such that $\Lambda(GHZ) = \psi_W$ for some state of the form ψ_W.*

Proof. Since $G_{\mathcal{FS}}(GHZ) = 1/2$, it suffices to find a state ψ_W such that $R_{\mathcal{FS}}(\psi_W) \leq 1$, which can be done since the robustness is continuous and there are states ψ_W arbitrarily close to the set of FS states. Then,

$$\Lambda(\eta) = \mathrm{tr}(GHZ\eta)\psi_W + \mathrm{tr}\left[(\mathbb{1} - GHZ)\eta\right]\tau_{\mathcal{FS}}, \tag{59}$$

where $\tau_{\mathcal{FS}} \in \mathcal{FS}$ is the state such that that $R_{\mathcal{FS}}(\psi_W) = R(\psi_W||\tau_{\mathcal{FS}})$, is the required map. □

III. BSP REGIME

Theorem 4. *In the resource theory of entanglement where the free operations are BSP maps, there exists a maximally GME state on every H. Namely, $\forall|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$, there exists a CPTP BSP map Λ such that $\Lambda(GHZ(n,d)) = \psi$.*

Proof. For every given $|\psi\rangle \in \left(\mathbb{C}^d\right)^{\otimes n}$, let

$$\Lambda(\eta) = \mathrm{tr}(\eta\, GHZ(n,d))\, \psi\, + \mathrm{tr}\left[(\mathbb{1} - GHZ(n,d))\eta\right] \rho_{\mathcal{BS}} \tag{60}$$

where $\rho_{\mathcal{BS}} \in \mathcal{BS}$ is the state which gives the (biseparable) robustness of ψ (i.e. $R_{\mathcal{BS}}(\psi) = R(\psi||\rho_{\mathcal{BS}})$). Then, $\Lambda(GHZ(n,d)) = \psi$ and it remains to be shown that Λ is BSP. As argued in the proofs of Theorems 1 and 2, this happens iff

$$R_{\mathcal{BS}}(\psi) \leq \frac{G_{\mathcal{BS}}(GHZ(n,d))}{1 - G_{\mathcal{BS}}(GHZ(n,d))}\,. \tag{61}$$

As discussed in the main text, it follows from equation (6) that $G_{\mathcal{BS}}(GHZ(n,d)) = (d-1)/d$ and, therefore, Λ is BSP iff $R_{\mathcal{BS}}(\psi) \leq d-1$. It is shown in [3] that for every bipartite pure state $\psi_{A|B}$ with Schmidt decomposition

$$\psi_{A|B} = \sum_i \sqrt{\lambda_i^{A|B}}\, |i\rangle_A\, |i\rangle_B \tag{62}$$

it holds that

$$R_{\mathcal{BS}}(\psi) = \left(\sum_i \sqrt{\lambda_i^{A|B}} \right)^2 - 1\,. \tag{63}$$

Thus

$$\begin{aligned} R_{\mathcal{BS}}(\psi) &\leq \min_{M \subsetneq [n]} \left(\sum_i \sqrt{\lambda_i^{M|\bar{M}}} \right)^2 - 1 \\ &\leq d-1 \end{aligned} \tag{64}$$

where the latter inequality follows from considering the state with all eigenvalues $\lambda_i = 1/d$. Hence, $\forall\,|\psi\rangle \in \left(\mathbb{C}^d\right)^{\otimes n}$ there exists a BSP map Λ such that $\Lambda(GHZ) = \psi$. □

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[18] This shows, in particular, that the robustness of all ψ^+_{GHZ} states is always less than or equal to 2.